Generalisation of modular form $\longleftrightarrow$ modular curves $\Gamma\backslash\mathcal{H}$

$\downarrow$ (analytics) automorphic forms $\qquad$ $\downarrow$ (geometry) Shimura varieties

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Nuts

<u>Modular form</u> $f: \mathcal{H} \to \mathbb{C}$. $f(\gamma z) = (cz+d)^k f(z)$ $\forall \gamma \in \Gamma$, $z \in \mathcal{H}$

weight $\qquad$ level

$SL_2(\mathbb{R}) \circlearrowright \mathcal{H}$ transitively

$\mathcal{H} \cong SL_2(\mathbb{R}).i \qquad \mathrm{Stab}_{SL_2(\mathbb{R})}(i) = SO_2(\mathbb{R})$

- cuspidal q-expansion has constant term 0 at every cusp.
- Hecke operators T_n ($n \in \mathbb{Z}_{\geq 0}$) related to $|_k \Gamma\alpha\Gamma$, $\alpha \in GL_2(\mathbb{Q})^{\det>0}$, $\det \alpha = n$. slash operator
- moduli interpretation of $\Gamma\backslash\mathcal{H}$ parametrises elliptic curves + level structure.

<u>Observation</u> $SL_2(\mathbb{R}) = Sp_2(\mathbb{R})$

Recall $Sp_{2n}(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(F) \;\middle|\; {}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$

$\Longleftrightarrow \begin{cases} {}^tAC,\ {}^tBD \text{ symmetric} \\ {}^tAD - {}^tCB = I_n \end{cases}$

We have seen the generalisation to $Sp_{2n}(\mathbb{R}) \circlearrowright \mathcal{H}_n$ Siegel upper half plane

Siegel modular forms

$\Gamma\backslash\mathcal{H}_n$ parametrises polarised abelian varieties + level structures

$\mathbb{C}/\Lambda \qquad \mathbb{C}^n/$ lattices

$SL_2(\mathbb{R}) \cong SU_{1,1}(\mathbb{R}) \longrightarrow U_{a,b}(\mathbb{R})$

- $SL_2(\mathbb{R}) \circlearrowright \mathcal{H}$ $\{\pm I\}$ acts trivially on $\mathcal{H}$.
- $PGL_2(\mathbb{R})^{\det>0} \circlearrowright \mathcal{H}$. $\longrightarrow SO_{a,b}(\mathbb{R})$

  $\wr\wr$ $\quad$ det

  $SO_{2,1}(\mathbb{R})$ $\quad$ spinor norm

<u>Generalise to all reductive groups</u>

<u>Defn</u> An algebraic group G/F is said to be <u>reductive</u> if $G_{\bar{F}}$ has trivial unipotent radical.

<u>Ex</u> SL_n, GL_n, $U_{a,b}$, Sp_{2n}, SO_n, $Spin_n$, PGL_n, GSp_{2n}, E_6, E_7, E_8.

<u>Step I</u> Go to groups (from $\mathcal{H}$ to $SL_2(\mathbb{R})$)

f modular form of weight ℓ and level Γ: $\mathcal{H} \to \mathbb{C}$

Define $F: SL_2(\mathbb{R}) \to \mathbb{C}$

$$g \longmapsto j(g,i)^{-\ell} f(g.i)$$

$j(g,z) := cz+d$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\hookrightarrow$ automorphic factor.

$j(g_1g_2, z) = j(g_1, g_2z)\, j(g_2, z)$ (1-cocycle)

Level Γ, $\gamma \in \Gamma$, $F(\gamma g) = j(\gamma g, i)^{-\ell} f(\gamma g.i)$

$= j(\gamma, gi)^{\ell} j(g,i)^{-\ell} j(\gamma, gi)^{-\ell} f(gi) = F(g)$.

$\downarrow$

invariant under left translation by Γ.

Weight ℓ $\quad k \in SO_2(\mathbb{R})$ $\rightsquigarrow$ right translation by $k(\theta)$ is given by scalar multiplication by $e^{-i\ell\theta}$.

$$k = k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{aligned} F(gk) &= j(gk.i)^{-\ell} f(gk.i) \\ &= \underbrace{j(g.ki)^{-\ell}}_{} \underbrace{j(k.i)^{-\ell}}_{} \underline{f(g.i)} \\ &= F(g)\, e^{-i\ell\theta} \end{aligned}$$

$$\begin{aligned} j(k.i) &= i\sin\theta + \cos\theta \\ &= e^{i\theta} \end{aligned}$$

$$SO_2(\mathbb{R}) \longrightarrow \mathbb{C}$$
$$k(\theta) \longmapsto e^{-i\ell\theta}$$

<u>Exercise</u> interpret holomorphicity and growth condition

(hint : related to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

q-expansion. cuspidality

Assume $\Gamma = SL_2(\mathbb{Z})$ for simplicity $f(z) = \sum_{n=0}^{\infty} a_n q^n$. $\quad q = e^{2\pi i z}$

$$a_n = \int_{\mathbb{Z}\backslash\mathbb{R}} f(z) e^{-2\pi i n z}\, dx \qquad (z = x+iy).$$

$z = g.i \qquad g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k \qquad (k \in SO_2(\mathbb{R}))$

Gram-Schmidt Iwasawa decomposition

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} k$$

<u>Ex</u> $\quad a_0 = \int_{\mathbb{Z}\backslash\mathbb{R}} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} k\right) y^{-1/2}\, dx$

Let $U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in SL_2 \right\}$

$$a_0 = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} F(u*)\, du$$

$$a_0 = 0 \Leftrightarrow \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} F(ug)\, du = 0 \quad \forall g \in SL_2(\mathbb{R})$$

<u>Step II</u> Adelisation

$$\left(\begin{array}{l} F: SL_2(\mathbb{R}) \longrightarrow \mathbb{C} \\ \quad\quad \downarrow \\ \Phi: GL_2(\mathbb{A}_{\mathbb{Q}}) \longrightarrow \mathbb{C} \end{array} \right)$$

$\leftarrow$ classical automorphic forms?